INDIAN INSTITUTE OF TECHNOLOGY MADRAS

BT3011

CFA EXERCISE

# Trajectory of a projectile fluid in laminar flow

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# 1 Introduction

Laminar flow occurs when a fluid flows in parallel layers, with no disruption between the layers. At low velocities, the fluid tends to flow without lateral mixing, and adjacent layers slide past one another. The motion of the particles of the fluid is very orderly with particles close to a solid surface moving in straight lines parallel to the surface. Laminar flow occurs at lower velocities, below a threshold after which it becomes turbulent. Because of all the smooth characteristics of laminar flow, very interesting properties are observed.

The general populace is accustomed to seeing the turbulent flow of water and as such is very intrigued by the smooth properties observed in laminar flow. Some engineers have exploited this and used laminar flow to set up stunning displays at hotspots like casinos and expensive hotels and such. One such example is a fountain which shoots jets of liquid in laminar flow so that it's trajectory can be predetermined to set up interesting displays. This article is going to talk about how one would be able to model the same trajectory for different fluids.

# 2 Background Knowledge

## 2.1 Engineering Bernoulli Equation

The Engineering Bernoulli equation can be considered to be a statement of the conservation of energy principle. This equation allows us to relate the change in external conditions between two points in a pipe flow[1].
Between two points in a pipe flow,
the Engineering Bernoulli equation becomes:

$$\frac{\Delta p}{\rho} + \frac{\Delta v^2}{2} + g\Delta x + \widehat{FL} + \widehat{W}_s = 0 \tag{2.1}$$

where $\widehat{FL}$ stands for frictional loss and $\widehat{W}_s$ stands for work done.

## 2.2 Projectile Motion

A point projectile experiences forces due to gravity and air friction. Air friction shall be ignored here to simplify calculations. When a projectile is shot from the ground at an elevation angle $\theta$ with an initial velocity $u$ the time of flight is found by :

$$-usin\theta = usin\theta - gt \tag{2.2}$$

$$t = \frac{2usin\theta}{g} \tag{2.3}$$

So for $0 \leq t \leq 2usin\theta$ :

$$v_y = usin\theta - gt \tag{2.4}$$

$$v_x = ucos\theta \tag{2.5}$$

The trajectory of the particle can be found out to be :

$$y = utsin\theta - \frac{gt^2}{2} \tag{2.6}$$

$$x = utcos\theta \tag{2.7}$$

## 2.3 Reynold's Number

Reynold's Number is a dimensionless quantity that is used to help predict similar flow patterns in different fluid flow situations. The Reynold's Number is defined as the ratio of inertial forces to viscous forces and consequently quantifies the relative importance of these two types of forces for given flow conditions. Here, Reynold's Number is being used to predict and regulate flow pattern of the fluid at any point and maintain it below the threshold to maintain laminar flow. In the case of pipe flow that we are dealing with, laminar flow prevails for Reynold's Number below 2100, while turbulent flow prevails for values above 4200. Mathematically, Reynold's Number for pipe flow becomes :

$$N_{Re} = \frac{\rho v D}{\mu} \tag{2.8}$$

where $\rho$ stands for density of fluid, $v$ for velocity of fluid, $d$ stands for diameter of pipe and $\mu$ stands for viscosity of fluid.

## 2.4 Frictional Loss

The Engineering Bernoulli equation includes a term for frictional losses at the pipe wall(skin friction) - $\widehat{FL}$ . In a pipe flow, it mathematically boils down to :

$$\widehat{FL} = (\frac{4fL}{D})\frac{v_{avg}^2}{2} \tag{2.9}$$

where $f$ is the friction factor, $L$ is length of the pipe, $v_{avg}$ is the average velocity in the pipe and $D$ is the diameter of the pipe.

# 3 Calculation of Trajectory

## 3.1 Assumptions and Requisites

### 3.1.1 Assumptions

- Air friction to the projectile motion is assumed to be negligible.
- Stream of water flowing out of pipe is considered to be made up on individual droplets small enough to be considered as independent projectiles with the same trajectory.
- Newtonian fluid with constant density and viscosity.

### 3.1.2 Requisites

- Pressure drop established at shooting site to counter effects of skin friction and gravity.
- Velocity of fluid at any point should be lower than the maximum velocity permitted for laminar flow in the system.

## 3.2 Formulation

The problem can be divided into three stages:

### 3.2.1 Shooting pipe

In the pipe which is at an inclination of $\theta$ to the ground, a pressure difference is supplied such that the velocity of the fluid is maintained constant throughout the pipe. Applying the Engineering Bernoulli equation to the above statement gives :

$$\frac{\Delta p}{\rho} + g\Delta z + \widehat{(}FL) = 0$$

$$\Delta p = -\rho(\frac{4fL}{D}\frac{v_{avg}^2}{2} + gLsin\theta) \tag{3.1}$$

where $L$ is the length of the shooting pipe.

### 3.2.2 Projectile Motion

Once the liquid has been shot out of the pipe, it's assumed that the stream of water behaves as a collection of individual point particles and that the air friction is negligible. The trajectory equation is as given in Eq. 2.6 and Eq. 2.7 where $u = v_{avg}$ from Eq. 3.1.

$$y = utsin\theta - \frac{gt^2}{2}$$

$$x = utcos\theta$$

Other useful relations are:
Maximum height of projectile:

$$h = \frac{u^2sin^2\theta}{2g} \tag{3.2}$$

Range of projectile:

$$S = \frac{2u^2cos\theta sin\theta}{g} \tag{3.3}$$

### 3.2.3 Limiting Factors

For our assumptions to be valid, the flow has to be laminar at all points. That means that the Reynold's Number must be less than 2100 at every point. To make calculations simpler, and to avoid turbulence at any cost, the limit on $N_{Re}$ will be rounded to 2000.

$$N_{Re} = \frac{\rho v D}{\mu} < 2000$$

$$vD < 2000\frac{\mu}{\rho} \tag{3.4}$$

This becomes the limiting product of velocity of fluid and diameter of pipe in terms of viscosity and density. Since viscosity and density both are constant for a given Newtonian fluid, this product is characteristic for a liquid flowing in a pipe.

## 3.3 Upper Bounds

As seen in Eq.3.4, for laminar flow:

$$vD < 2000\frac{\mu}{\rho}$$

If we were to keep the diameter of the pipe $D$ constant, then we can calculate the upper bounds for the velocity of the liquid for a given liquid.

$$v_{max} = 2000\frac{\mu}{\rho D} \tag{3.5}$$

From Eq.3.1,

$$\Delta p = -\rho(\frac{4fLv_{max}^2}{2D} + gLsin\theta)$$

for a given L and where :

$$f = \frac{16}{N_{Re}} = \frac{16}{2000}$$

which becomes, pressure drop needed in case of maximum velocity :

$$\Delta p_{max} = -\rho(64000\frac{\mu^2}{\rho^2 D^3} + gLsin\theta) \quad (3.6)$$

For maximum range of projectile, $\theta = \frac{\pi}{4}$ :

$$S_{max} = (2000)^2 \frac{\mu}{\rho D g} \quad (3.7)$$

## 3.4 $v_{max}$ for different fluids

$D$ is taken to be fixed at 5cm. Kinematic viscosity $\nu$ for some liquids were obtained[2] and their respective $v_{max}$ were calculated.

Table 3.1: $v_{max}$ for different fluids

| Liquid | $\nu = \frac{\mu}{\rho}$(in $m^2 s^{-1}$) | $v_{max} = 2000\frac{\nu}{D}$ (in $ms^{-1}$) |
|---|---|---|
| Distilled Water(16°C) | $1.0038x10^{-6}$ | $4x10^{-2}$ |
| Sea Water | $1.15x10^{-6}$ | $4.6x10^{-2}$ |
| Glycerine 100%(20°C | $648x10^{-6}$ | 25.92 |
| Vinegar(15°C) | $1.35x10^{-6}$ | $5.4x10^{-2}$ |
| Corn Starch Solution(22 Baume 21°C) | $32.1x10^{-6}$ | 1.284 |

# 4 Possible Venture

The model could be taken one step further by using the shooting pipe only to focus the flow into a laminar one and by choosing initial velocity at the entry of the pipe such that the loss in velocity till the exit of the pipe compensates for the skin friction and gravity. Calculations in such cases tend to complicate further as both Reynold's Number and skin friction loss are dependent on the velocity at individual points. Due to the complexity of the resulting Engineering Bernoulli equation, it is not discussed here.

# Acknowledgement

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## References

[1] *Suraishkumar GK. 2014. Continuum Analysis of Biological Systems: Conserved Quantities, Forces and Fluxes.*

[2] *http://www.engineeringtoolbox.com/*